Chapter 2: Derivatives!

You fire a projectile into the air. Its position above its starting point is given by

$$x(t) = t(3-t) = 3t - t^2$$

(x in kilometers, t in seconds)

Q. How do you find the average velocity from $t=1$ to $t=2$?

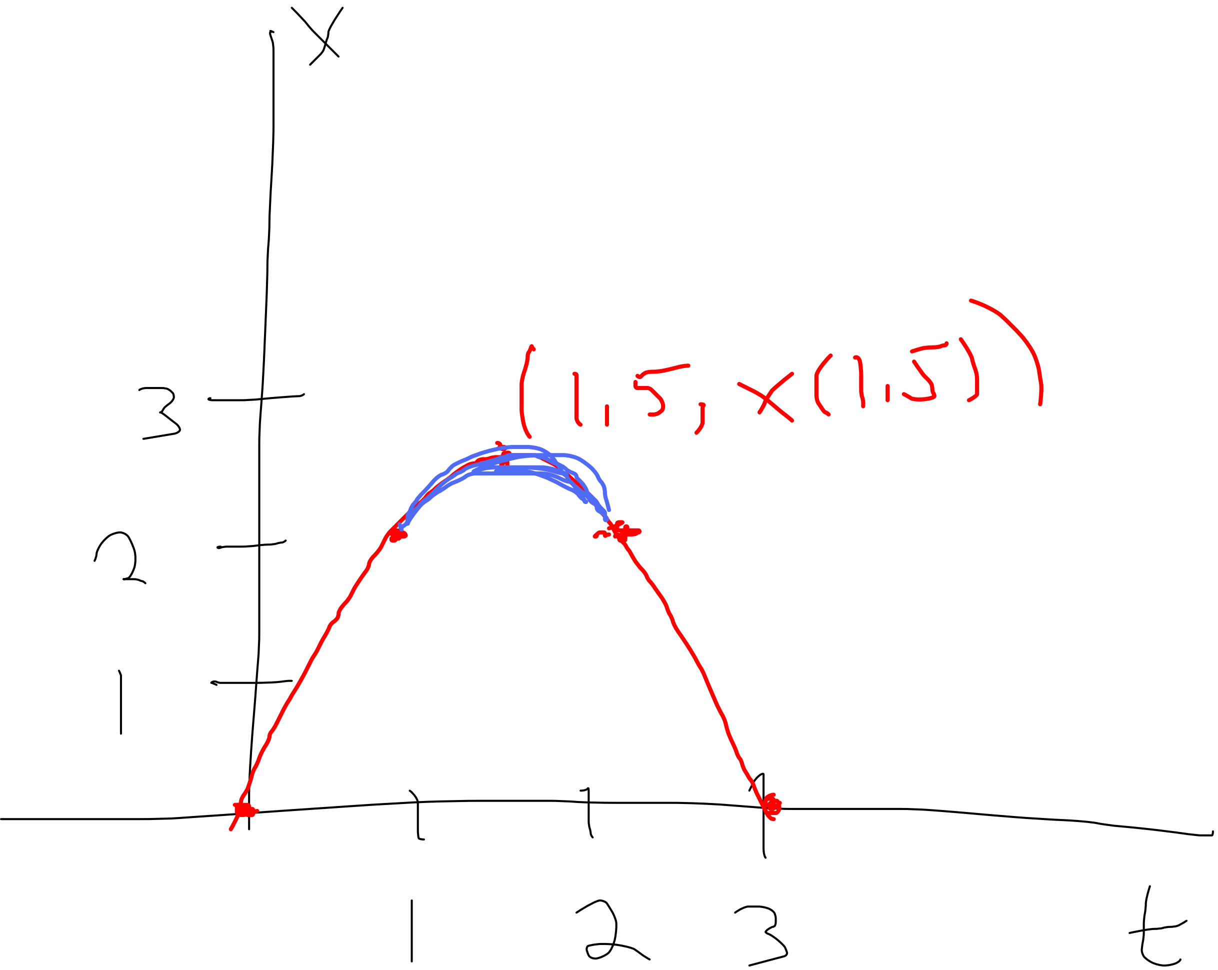
$$\text{Velocity} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{x(2) - x(1)}{2-1}$$

$$= \frac{2(3-2) - 1(3-1)}{2-1}$$

$$= \frac{2 \cdot 1 - 1 \cdot 2}{1} = 0$$

Picture



How about the average velocity from $t=1$ to an arbitrary time $t=a$?

$(0 \leq a \leq 3)$

$$\text{Velocity} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{x(1) - x(a)}{1 - a}$$

$$= \frac{2 - x(a)}{1 - a}$$

Finally, how about from $t = t_0$ to an arbitrary point $t = a$?

$$\text{Velocity} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{x(t_0) - x(a)}{t_0 - a}$$

$(0 \le t_0 \le 3, \quad 0 \le a \le 3)$

We have problems if $a = t_0$. Then we always get $\frac{0}{0}$, more work!

We still want to interpret as a kind of velocity — instantaneous velocity!

The $\boxed{\text{derivative}}$ of x at time $t = a$ is the instantaneous velocity. It is defined at

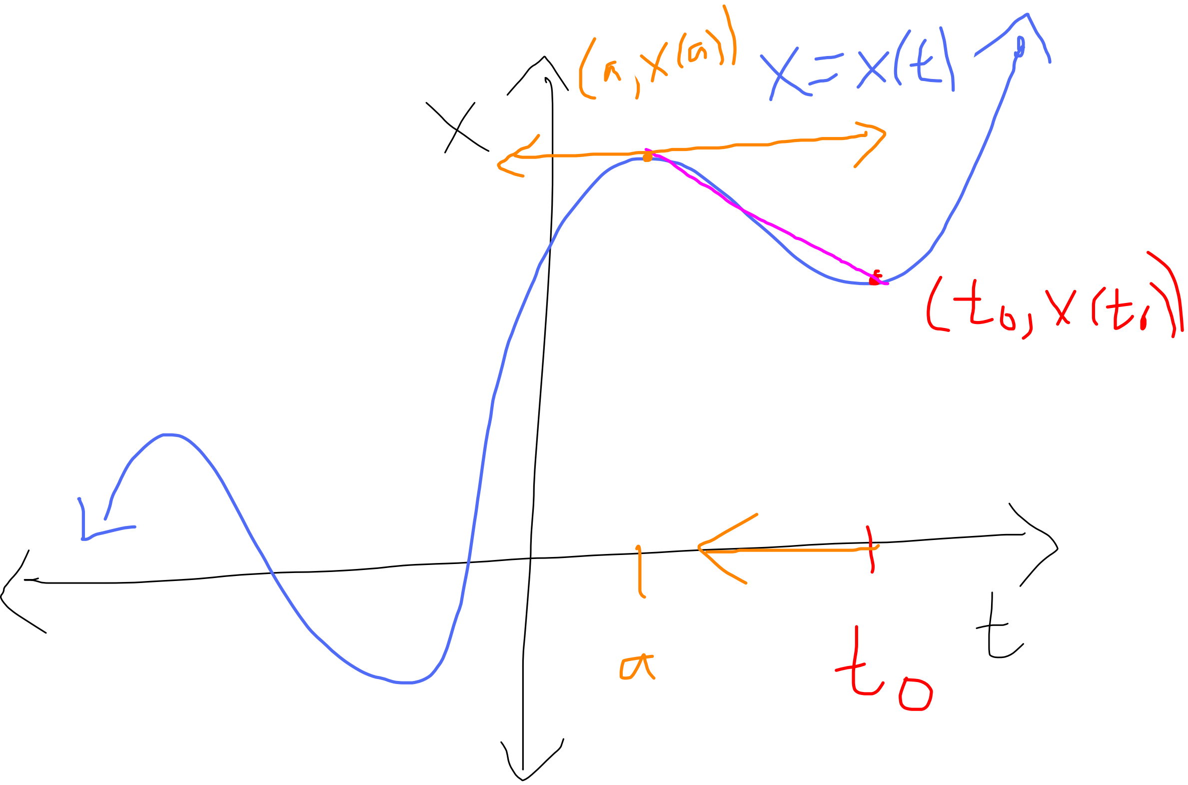
$$\boxed{\lim_{t \to a} \frac{x(t) - x(a)}{t - a}} \; *$$

and denoted by $\boxed{x'(a)}$.

1st take derivative, then plug in a.

* provided the limit exists!

Picture:



Purple line = secant line

Slide t_0 over to a to get a tangent line.

Slope of secant line = avg. velocity

Slope of tangent line = ins. velocity

Warning: you have to take limits from the right **and** left and have them equal eachother in order for $x'(a)$ to exist.

(watch out for absolute values)

Example: Find $x'(1)$

Here, $a = 1$.

We want

$$\lim_{t \to 1} \frac{x(t) - x(1)}{t - 1}$$

(always plug in the value for a if you are given it)

$$\frac{x(t) - x(1)}{t-1}$$

$$= x(1)$$

$$= \frac{3t - t^2 - 2}{t-1}$$

$$= \frac{\cancel{(t-1)}(-t+2)}{\cancel{t-1}}$$

$$= -t+2 \quad \text{if } t \neq 1.$$

Take limit.

So

$$x'(1) = \lim_{t \to 1} \frac{x(t) - x(1)}{t-1}$$

Keep limit!

$$= \lim_{t \to 1} (-t+2)$$

$$= -1+2 = \boxed{1}$$

General Definition: (derivative)

Let f be a function,

let $x=a$ be a point. Then

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$$

provided the limit exists.

If the limit exists, we say that f is **differentiable** at $x=a$.

By substituting $h = x-a$,

we arrive at

$$h \qquad\qquad = 0$$

$$\frac{f(x-a+a)-f(a)}{x-a} \quad = h$$

$$= \frac{f(a+h)-f(a)}{h}$$

Change a to x (abusively)

to get

$$\frac{f(x+h)-f(x)}{h}$$

We had

$\lim_{x \to a}$, but this is

the same as $\lim_{(x-a) \to 0}$.

Since $x - a = h$, we get

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

This function is called the

derivative of f.

Example 2: $f(x) = \sqrt{x-2}$

Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h}$$

Plug in $h=0$, get $\frac{0}{0}$ = more work!

Multiply by the conjugate,

Do not distribute the

h on the denominator.

$$\frac{\sqrt{x+h-2}-\sqrt{x-2}}{h}\cdot\frac{(\sqrt{x+h-2}+\sqrt{x-2})}{(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$=\frac{(\sqrt{x+h-2})^2-(\sqrt{x-2})^2}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$=\frac{\cancel{x}+h-\cancel{2}-\cancel{x}+\cancel{2}}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$

Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-2} + \sqrt{x-2}\right)} \right)$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$$= \boxed{\frac{1}{2\sqrt{x-2}}} \quad \text{(plug in } h=0\text{)}$$

Note: $x=2$ is not in the domain of $f'(x)$, but is in the domain of $f(x)$.

Example 3: Does $f'(0)$ exist for $f(x) = |x|$?

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x|}{x}$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$f'(0)$ does not exist!